Riemannian Geometry $\boxed{\text{Week 3}}$

Tangent and cotangent bundle

Let M be a smooth manifold and $p \in M$.

T_pM: tangent space at p, T_p^*M : cotangent space.

<u>Def.</u> Tangent bundle:

$$TM = \bigcup_{p \in M} T_pM$$

Projection: $\pi : TM \to M$, $T_pM \xrightarrow{\pi} p$

It naturally has the structure of a <u>smooth manifold</u> of dimension $\dim(TM) = 2\dim(M)$:

If (U, ϕ) is a local coordinate chart around $p \in M$, with corresponding coordinates $(x^1, \dots, x^n)$

Local coordinate chart on $TU = \bigcup_{p \in U} T_pM$:

$$\tilde{\varphi} : TU \longrightarrow \mathbb{R}^n \times \mathbb{R}^n,$$

$$\tilde{\varphi}(p, \underbrace{v}_{v \in T_pM}) = (x^1(p), \dots, x^n(p) \,;\, v^1, \dots, v^n)$$

$v = v^i \frac{\partial}{\partial x^i}$, so $v^i = dx^i(v)$

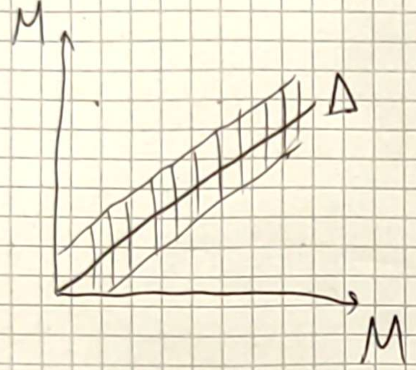
So locally: TM looks like the product $U \times \mathbb{R}^n$

But not necessarily globally!

<u>Examples</u>: $M = S^1$, then $TS^1 \cong S^1 \times \mathbb{R}$

But TS^2 is <u>not</u> a product bundle

<u>Note</u>: As a manifold, TM is always homeomorphic to an open neighborhood of the diagonal Δ in $M \times M$



(given a Riemannian metric, this identification follows using the exponential map).

So: For any two differentiable structures on M, the corresponding tangent bundles are homeomorphic as manifolds (same homotopy invariants).

Similarly for T^*M.

A <u>vector field</u> X: Smooth <u>section</u> of TM

i.e. $X : M \longrightarrow TM, \quad p \longrightarrow X_p \in T_pM$

$$\Pi \circ X = id_M.$$

Set of vector fields:

$\Gamma(TM)$ or $\Gamma(M)$.

$\Gamma(M)$ is a $C^\infty(M)$ module: If $\left.\begin{matrix} f \in C^\infty(M), \\ X \in \Gamma(M), \end{matrix}\right\} f \cdot X \in \Gamma(M)$

Similarly: Space of 1-forms: $\Gamma(T^*M)$ or $\Gamma^*(M)$

<u>Example</u>: If $f_1, \dots, f_s, h_1, \dots, h_s \in C^\infty(M)$,

$$\omega = \sum_{i=1}^{s} f_i \, dh_i \ : \text{ is an 1-form}$$

<u>Remark</u>: If $X \in \Gamma(M)$, $X|_p \neq 0$: Locally I can find coordinates around p such that $X = \frac{\partial}{\partial x^1}$.

If $\omega \in \Gamma^*(M)$: In general, I <u>cannot</u> express ω as df for some $f \in C^\infty(M)$ (unless $d\omega = 0$).

<u>Tensors at a point p</u>:

Let $\omega \in T_p^*M$. I can think of it as a linear map

$$\omega : T_pM \longrightarrow \mathbb{R}.$$

Similarly: I can think of any $X \in T_pM$ as

$$X : T_p^*M \longrightarrow \mathbb{R}$$

<u>Definition</u>: A tensor of type (k, l) at p is a <u>multilinear</u> map

$$T : \underbrace{T_p^*M \times \dots \times T_p^*M}_{k \text{ copies}} \times \underbrace{T_pM \times \dots \times T_pM}_{l \text{ copies}} \longrightarrow \mathbb{R}.$$

<u>Example</u>:
- Tangent vector: $(1,0)$
- Covector: $(0,1)$
- Riem. metric g: $(0,2)$

Tensor product:

If T_1 of type (k_1, l_1), T_2 of type (k_2, l_2),

$T_1 \otimes T_2$: of type (k_1+k_2, l_1+l_2)

covectors → ← vectors

$$T_1 \otimes T_2 (\omega_1, \dots, \omega_{k_1}, \omega_{k_1+1}, \dots, \omega_{k_1+k_2}; v_1, \dots, v_{l_1}, v_{l_1+1}, \dots, v_{l_1+l_2})$$

$$:= T_1(\omega_1, \dots, \omega_{k_1}; v_1, \dots, v_{l_1}) \cdot T_2(\omega_{k_1+1}, \dots, \omega_{k_1+k_2}; v_{l_1+1}, \dots, v_{l_1+l_2})$$

Note: $T_1 \otimes T_2 \neq T_2 \otimes T_1$. And Multilinear:
$T_1 \otimes (\lambda T_2) = \lambda \cdot (T_1 \otimes T_2)$, $\lambda \in \mathbb{R}$

Coordinate basis:

If $(x^1, \dots, x^n)$ local coordinates around p:

$\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^n$: Basis of T_pM, $\{dx^i\}_{i=1}^n$: Basis of T_p^*M

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l} \right\}_{i_1, \dots, i_k, j_1, \dots, j_l = 1}^n$$: Basis of (k,l) - tensors

And

$$T = T^{i_1 \dots i_n}_{j_1 \dots j_l} \cdot \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_n}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_n}$$

That's why we have been writing $g = g_{ij}\, dx^i \otimes dx^j$

Change of coordinates:

$$(x^1, \dots, x^n) \longrightarrow (y^1, \dots, y^n)$$

$$dy^i = \frac{\partial y^i}{\partial x^a} dx^a$$

$$\frac{\partial}{\partial y^j} = \frac{\partial x^\beta}{\partial y^j} \frac{\partial}{\partial x^\beta}$$

So: $$\boxed{\tilde{T}^{i_1 \dots i_k}_{j_1 \dots j_l} = T^{a_1 \dots a_k}_{\beta_1 \dots \beta_l} \cdot \frac{\partial y^{i_1}}{\partial x^{a_1}} \cdot \dots \cdot \frac{\partial y^{i_k}}{\partial x^{a_k}} \cdot \frac{\partial x^{\beta_1}}{\partial y^{j_1}} \cdot \dots \cdot \frac{\partial x^{\beta_l}}{\partial y^{j_l}}}$$

- Covariant part: Indices down (transforms with multiplication with $\frac{\partial x}{\partial y}$)
- Contravariant part: Indices up (transforms with $\frac{\partial y}{\partial x}$).

Space of tensors of order (k,l) over p:

$$\otimes^k T_pM \otimes^l T_p^*M$$

Note: If $Q \in \otimes^k T_pM \otimes^l T_p^*M$:

$$Q: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{l} \longrightarrow \mathbb{R}$$

<u>Remark:</u> If T is a (k,l) tensor over a vector space V:

$$T: \underbrace{V^* \times \dots \times V^*}_{k} \times \underbrace{V \times \dots \times V}_{l} \longrightarrow \mathbb{R}$$

Then:

If I "forget" (k_1, l_1) of its entries:

I can think of it a map

$$T: \underbrace{V^* \times \dots \times V^*}_{k-k_1} \times \underbrace{V \times \dots \times V}_{l-l_1} \longrightarrow \otimes^{k_1} V \otimes^{l_1} V^*$$

i.e a multilinear map in the rest of the (k_1, l_1)-variables

In this way:

I can identify for example,

$$Q \in V \otimes V^* \quad \text{with a map} \quad Q: V \longrightarrow V$$

$(1,1)$ tensor

<u>Tensor product in coordinates</u>:

$$(f \otimes g)^{i_1 \dots i_{k_1} i_{k_1+1} \dots i_{k_1+k_2}}_{j_1 \dots j_{l_1} j_{l_1+1} \dots j_{l_1+l_2}} = f^{i_1 \dots i_{k_1}}_{j_1 \dots j_{l_1}} \cdot g^{i_{k_1+1} \dots i_{k_1+k_2}}_{j_{l_1+1} \dots j_{l_1+l_2}}$$

<u>Another operation on tensors</u>: Contraction

Choose one contravariant and one covariant index and some them

$$tr_{(i,j)}: \{(k,l) \text{ tensors}\} \longrightarrow \{(k-1, l-1) \text{ tensors}\}$$

with respect to the i-th index up, j-th index down

Example: $\left(tr_{(1,1)} T\right)^{i_2 \cdots i_k}_{j_2 \cdots j_l} = T^{a i_2 \cdots i_k}_{a j_2 \cdots j_l}$

I will denote it simply as tr when no confusion arises.

In the case of a $(1,1)$ tensor: This is just its trace as a linear map.

It is coordinate independent! (like the trace of a map)

See Exercises...

(As in the case of the usual trace, there is a coordinate-free definition, but not useful for actual computations)

Tensor field of order (k,l): An assignment at every $p \in M$ of a (k,l) tensor, with smooth components in any local coordinate system.

Note: T is a (k,l)-tensor field, then

$$T : \underbrace{\Gamma^*(M) \times \cdots \times \Gamma^*(M)}_{k} \times \underbrace{\Gamma(M) \times \cdots \times \Gamma(M)}_{l} \longrightarrow C^\infty(M)$$

If $\omega_1, \dots, \omega_k \in \Gamma^*(M)$, $X_1, \dots, X_\ell \in \Gamma(M)$

$T(\omega_1, \dots, \omega_k; X_1, \dots, X_\ell)|_p$ depends only

on $\omega_1|_p, \dots, \omega_k|_p$
$X_1|_p, \dots, X_\ell|_p$

In particular: If $X_1|_p = 0$,

$$T(\omega_1, \dots, \omega_k; X_1, \dots, X_\ell)|_p = 0.$$

So: T is $C^\infty(M)$ - multilinear in its arguments.

Ex. On $\mathbb{R}^n$ the map $D: \Gamma(\mathbb{R}^n) \times \Gamma(\mathbb{R}^n) \longrightarrow \Gamma(\mathbb{R}^n)$

$$D(v, X) = D_v X$$

is not $C^\infty(M)$-multilinear, so it cannot be a $(1,2)$ - tensor field.

Theorem: A map $T: \underbrace{\Gamma^*(M) \times \dots \times \Gamma^*(M)}_{k} \times \underbrace{\Gamma(M) \times \dots \times \Gamma(M)}_{\ell} \longrightarrow C^\infty(M)$

is a tensor field if and only if it is $C^\infty(M)$-multilinear.

Proof: In the exercises.

Vector Fields and flows

Recall: A vector field is a diff. operator: $X: C^\infty(M) \to C^\infty(M)$

In local coord. $X = X^i \frac{\partial}{\partial x^i}$

If $X, Y \in \Gamma(M)$: $X \cdot Y$: 2^{nd} order diff. operator

But $[X,Y] = X \cdot Y - Y \cdot X$ is a vector field!

Coordinate expression:

$$[X,Y]^k = X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i}$$

$[\cdot,\cdot]$: Commutator.

Properties:

- $[X,Y] = -[Y,X]$ (anti-commutativity).
- $\mathbb{R}$-linearity
- Jacobi's identity

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

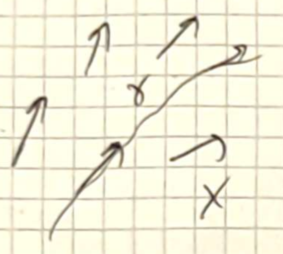
- $[f \cdot X, g \cdot Y] = f \cdot g\,[X,Y] + X(g) \cdot f \cdot Y - Y(f) \cdot g \cdot X$

So $[\cdot,\cdot] : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ is not tensorial.

$(\Gamma(M), [\cdot,\cdot])$: Is a Lie algebra. Of which Lie group? We will see later.

Flow of a vector field

Let $X \in \Gamma(M)$. A curve $\gamma : (a,b) \longrightarrow M$ is an integral curve of X if $\dot{\gamma}(t) = X|_{\gamma(t)} \quad \forall\, t \in (a,b)$



In local coordinates:

$$\dot{\gamma}^k(t) = X^k(\gamma(t)).$$

From the theory of ODEs:

The initial value problem $\begin{cases} \frac{du}{dt} = F(t, u(t)) \\ u(0) = u_0 \end{cases}$, $u \in \mathbb{R}^N$, $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is smooth

- Has a unique solution on a maximal interval $(-T_-, T_+) \subseteq \mathbb{R}$
- Smooth dependence on the initial data
- If $T_+ < +\infty$, then as $t \to T_+$: $|u(t)| \longrightarrow +\infty$ (and similarly for T_-)